

New Approximation of Logarithmic Mean

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Summary

I found a new approximation of logarithmic mean, which consists of addition, multiplication, division, and a square root. As the approximation is simple but accurate, it has many applications in education at college of technology and other schools. Some examples will be shown in this paper.

Key words: logarithmic mean, Shannon entropy, heat exchanger, pH

Introduction

For any positive real a, b , we define a mean $M(a, b)$ such that

$$M(a, b) = \frac{\frac{a+b}{2} + 2\sqrt{ab}}{3} = \frac{a+b+4\sqrt{ab}}{6}$$

Obviously

$$\sqrt{ab} \leq M(a, b) \leq \frac{a+b}{2}$$

and for any positive real k

$$M(ka, kb) = k \cdot M(a, b)$$

holds. The arithmetic-logarithmic-geometric mean inequality states that

$$\sqrt{ab} \leq \frac{a-b}{\ln a - \ln b} \leq \frac{a+b}{2}$$

Let's compare $\frac{a-b}{\ln a - \ln b}$ and $M(a, b)$. By the Maclaurin expansions

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

and

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$$

we have

$$\ln(x+1) \cdot (x+2+4\sqrt{x+1}) = 6x + \frac{x^5}{480} - \frac{x^6}{240} + \dots$$

So

$$\ln(x+1) \cdot (x+2+4\sqrt{x+1}) \sim 6x$$

$$\ln x \cdot (x+1+4\sqrt{x}) \sim 6(x-1)$$

$$\therefore \ln x \sim \frac{6}{x+1+4\sqrt{x}}(x-1)$$

Now we have a pretty good approximation such as

$$\ln x \sim \frac{x-1}{M(x,1)}$$

Both graphs are so close that it is very difficult to distinguish them. See Figure 1 below.

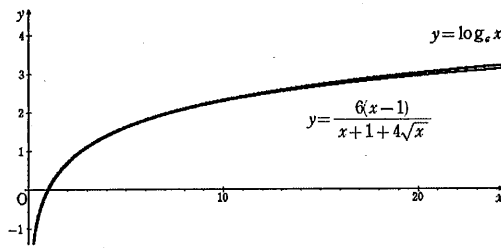


Figure 1. the natural logarithm and it's approximation

Let x be $\frac{a}{b}$. Then

$$\ln \frac{a}{b} \sim \frac{\frac{a}{b} - 1}{M(\frac{a}{b}, 1)} = \frac{b(\frac{a}{b} - 1)}{b \cdot M(\frac{a}{b}, 1)} = \frac{a-b}{M(a,b)}$$

So we have

$$\ln a - \ln b \sim \frac{a-b}{M(a,b)}$$

This implies

$$M(a,b) \sim \frac{a-b}{\ln a - \ln b}$$

Thus $M(a,b)$ is a new approximation of the logarithmic mean of a and b .

Approximation of the Binary Entropy Function

Let p, q be real numbers which satisfy the conditions

$$0 \leq p, q \leq 1, p + q = 1$$

and let H_e, H_2 denote

$$\begin{aligned} H_e(p, q) &= -p \ln p - q \ln q \\ H_2(p, q) &= -p \log_2 p - q \log_2 q \end{aligned}$$

H_2 is the binary entropy function and

$$H_2 = (\log_2 e) \cdot H_e \sim \sqrt{2} H_e$$

holds. By the new approximation $M(a, b)$,

$$H_e \sim -p \frac{p-1}{M(p,1)} - q \frac{q-1}{M(q,1)} = pq \left\{ \frac{1}{M(p,1)} + \frac{1}{M(q,1)} \right\}$$

To make this approximation much simpler, we substitute $M(1, p)$ by \sqrt{p} and $M(1, q)$ by \sqrt{q} . Then we have

$$H_e \sim pq \left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}} \right) = \sqrt{pq} (\sqrt{p} + \sqrt{q})$$

Both graphs of $y = \sqrt{x} + \sqrt{1-x}$ and $y = \frac{1}{\sqrt{x^2 + (1-x)^2}}$ are so close as you can see in

Figure 2, $\sqrt{p} + \sqrt{q}$ can be approximated by $\frac{1}{\sqrt{p^2 + q^2}}$.

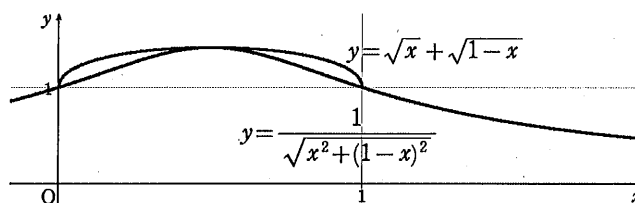


Figure 2. an approximation of $\sqrt{x} + \sqrt{1-x}$

Now we have

$$\sqrt{pq} (\sqrt{p} + \sqrt{q}) \sim \sqrt{\frac{pq}{p^2 + q^2}}$$

Let E be $\frac{pq}{p^2 + q^2}$, then we have

$$H_2 \sim \sqrt{2} H_e \sim \sqrt{2E}$$

Both graphs of H_2 and $\sqrt{2E}$ are shown in Figure 3. H_2 is expressed by a dotted line.

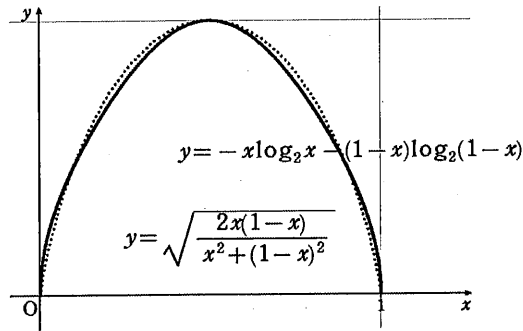


Figure 3. binary entropy function and it's approximation

As the channel capacity C of the binary symmetric channel equals to $1 - H_2$,

$$C \sim 1 - \sqrt{2E} = 1 - \sqrt{\frac{2p(1-p)}{p^2 + (1-p)^2}}$$

Let $p = \frac{1}{2}$, then we have $C = 0$.

Jianhua Lin proved the following inequality ([1]).

$$H_2(p, 1-p) \leq 2\sqrt{p(1-p)}$$

His proof needs knowledge of calculus, but the following proof is much easier.

For any real p ,

$$p^2 + (1-p)^2 = 2\left(p - \frac{1}{2}\right)^2 + \frac{1}{2} \geq \frac{1}{2}$$

holds. This implies

$$\frac{1}{p^2 + (1-p)^2} \leq 2$$

So we have

$$\sqrt{2E} = \sqrt{\frac{2p(1-p)}{p^2 + (1-p)^2}} \leq 2\sqrt{p(1-p)}$$

Now we can explain the inequality

$$H_2(p, 1-p) \leq 2\sqrt{p(1-p)}$$

to students without knowledge of calculus.

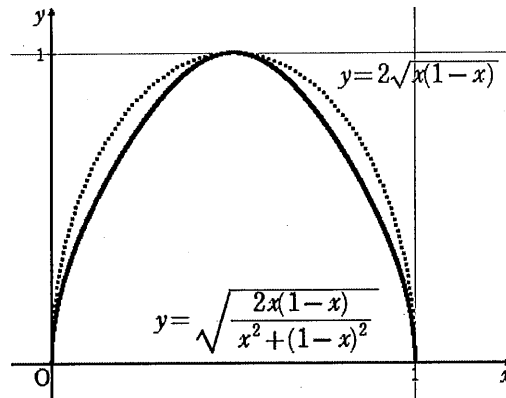


Figure 4. an upper bound for the binary entropy function

Basic Flow Arrangement in Tube Heat Exchangers

The most common arrangements for flow paths within a heat exchanger are counter flow and parallel flow. A parallel flow heat exchanger is one in which the directions of the flows are the same. A counter flow heat exchanger is one in which the direction of a flow is opposite to the other one. A parallel flow arrangement is illustrated left in Figure 5, and a counter flow arrangement is illustrated right in Figure 5.

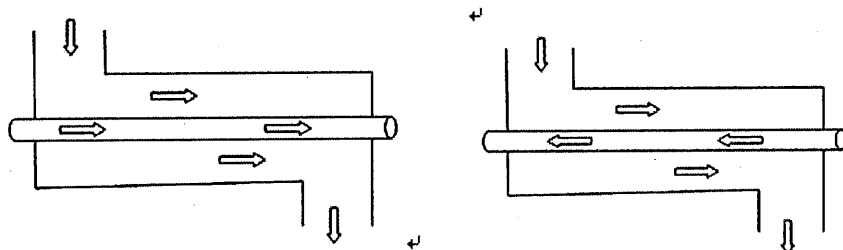


Figure 5. The basic flow arrangements

Under comparable conditions, more heat is transferred in a counter flow arrangement than in a parallel flow heat exchanger. An easy proof is given below.

Suppose that hot fluid temperature decrease from T_1 (K) down to T_2 (K), and cold fluid temperature increase from t_1 (K) up to t_2 (K). Then the following equation holds.

$$Q = UA \cdot \Delta T$$

where

- Q (W) is the amount of heat transferred,
- U (W/mK) is the overall heat transfer coefficient,
- A (m²) is the area for heat transfer,
- ΔT (K) is the logarithmic mean temperature difference.

In the counter flow arrangement, the logarithmic mean temperature difference (abbreviated as LMTD) is

$$\Delta T = \frac{(T_1 - t_2) - (T_2 - t_1)}{\ln(T_1 - t_2) - \ln(T_2 - t_1)} \quad (K)$$

On the other hand, in the parallel flow arrangement LMTD is

$$\Delta T = \frac{(T_1 - t_1) - (T_2 - t_2)}{\ln(T_1 - t_1) - \ln(T_2 - t_2)} \quad (K)$$

In each case the new approximation

$$M(a,b) \sim \frac{a-b}{\ln a - \ln b}$$

is extremely useful.

In the counter flow arrangement, LMTD is approximated as

$$\begin{aligned} \frac{(T_1 - t_2) - (T_2 - t_1)}{\ln(T_1 - t_2) - \ln(T_2 - t_1)} &\sim M(T_1 - t_2, T_2 - t_1) \\ &= \frac{T_1 + T_2 - t_1 - t_2 + 4\sqrt{(T_1 - t_2)(T_2 - t_1)}}{6} \end{aligned}$$

and in the parallel flow arrangement, LMTD is approximated as

$$\begin{aligned} \frac{(T_1 - t_1) - (T_2 - t_2)}{\ln(T_1 - t_1) - \ln(T_2 - t_2)} &\sim M(T_1 - t_1, T_2 - t_2) \\ &= \frac{T_1 + T_2 - t_1 - t_2 + 4\sqrt{(T_1 - t_1)(T_2 - t_2)}}{6} \end{aligned}$$

As they have the same expression $T_1 + T_2 - t_1 - t_2$ in numerators, we just compare $(T_1 - t_2)(T_2 - t_1)$ with $(T_1 - t_1)(T_2 - t_2)$.

As $t_1 < t_2$ and $T_1 > T_2$,

$$(T_1 - t_2)(T_2 - t_1) - (T_1 - t_1)(T_2 - t_2) = (t_2 - t_1)(T_1 - T_2) > 0$$

So we have

$$(T_1 - t_2)(T_2 - t_1) > (T_1 - t_1)(T_2 - t_2)$$

This implies

$$M(T_1 - t_2, T_2 - t_1) > M(T_1 - t_1, T_2 - t_2)$$

Thus LMTD is always greater in the counter flow arrangement.

Insulations of tube heat exchangers

For any positive real a, b , Let $HM(a, b)$ denote the harmonic mean of a, b , i.e.,

$$HM(a, b) = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

If $\lambda_1 < \lambda_2$ and $a < b$, then

$$(\lambda_1 a + \lambda_2 b) - (\lambda_2 a + \lambda_1 b) = (\lambda_2 - \lambda_1)(b - a) > 0$$

This implies

$$\lambda_1 a + \lambda_2 b > \lambda_2 a + \lambda_1 b$$

$$\frac{1}{\lambda_1 a + \lambda_2 b} < \frac{1}{\lambda_2 a + \lambda_1 b}$$

$$\frac{\lambda_1 \lambda_2 ab}{\lambda_1 a + \lambda_2 b} < \frac{\lambda_1 \lambda_2 ab}{\lambda_2 a + \lambda_1 b}$$

$$\therefore HM(\lambda_1 a, \lambda_2 b) < HM(\lambda_2 a, \lambda_1 b)$$

Suppose that we have two sort of thermal insulations A_1 and A_2 . Let the thermal conductivities be λ_1 and λ_2 respectively.

We cover a pipe with radius r by these insulations. And suppose that these insulations have the same thickness d . There are two possible ways. One way is to cover with A_1 first, and with A_2 next as illustrated left in Figure 6. The other way is to cover with A_2 first, and with A_1 next as illustrated right in Figure 6.

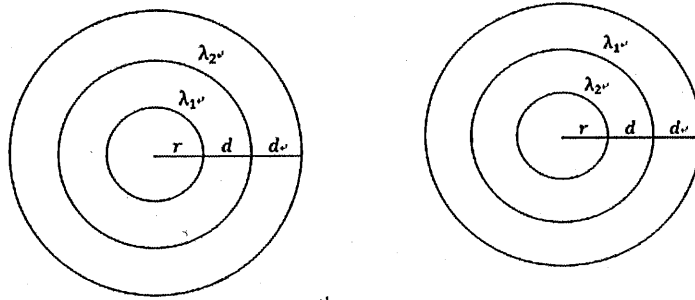


Figure 6. Two types of insulations

Suppose that $\lambda_1 < \lambda_2$. Which way is better to make heat loss decrease ?

Let ΔT be the difference between pipe's surface temperature and air temperature and insulation A_i is inside and A_j is outside. Then the total heat loss $H(W)$ per unit length can be calculated by

$$H = \frac{2 \pi \Delta T}{\frac{1}{\lambda_i} \ln \frac{r+d}{r} + \frac{1}{\lambda_j} \ln \frac{r+2d}{r+d}} (W).$$

By the approximation of the natural logarithm,

$$\begin{aligned}
 H &\sim \frac{2 \pi \Delta T}{\frac{1}{\lambda_i} \cdot \frac{d}{M(r,r+d)} + \frac{1}{\lambda_j} \cdot \frac{d}{M(r+d,r+2d)}} \\
 &= \frac{\pi \Delta T}{d} \cdot \frac{1}{\frac{1}{\lambda_i} \cdot \frac{1}{M(r,r+d)} + \frac{1}{\lambda_j} \cdot \frac{1}{M(r+d,r+2d)}} \\
 &= \frac{\pi \Delta T}{d} \cdot HM(\lambda_i M(r, r + d), \lambda_j M(r + d, r + 2d))
 \end{aligned}$$

As

$$M(r,r+d) < r+d < M(r+d,r+2d)$$

and $\lambda_1 < \lambda_2$, we have

$$HM(\lambda_1 M(r, r + d), \lambda_2 M(r + d, r + 2d)) < HM(\lambda_2 M(r, r + d), \lambda_1 M(r + d, r + 2d))$$

To make heat loss much decrease, we should cover pipe with insulations A_1 first, and with A_2 next.

Buffer System and New Approximation of pH

A pH number is defined as the negative base 10 logarithm of the hydrogen ion concentration. The equation is the following :

$$pH = -\log_{10} [H^+]$$

The unit of the hydrogen ion concentration is mol/l. If we use nanomol/l, the equation is :

$$pH = -\log_{10} \frac{[H^+]}{10^9} = 9 - \log_{10} [H^+]$$

In the human body, normal arterial blood pH is 7.4. In this case,

$$7.4 = 9 - \log_{10} [H^+]$$

$$\log_{10} [H^+] = 9 - 7.4 = 1.6$$

$$\therefore [H^+] = 10^{1.6} = 40 \text{ (nanomol/l)}$$

The following is a list of hydrogen ion concentrations and their pH's.

Table 1. hydrogen ion concentrations and their pH's

[H ⁺]	100	80	64	50	40	32	25	20	16
pH	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8

The following approximation of pH is widely known as "The Rule of 80"([2]).

$$pH = 7 + \frac{80 - [H^+]}{100}$$

But this approximation is not accurate as you can see by Table 2 below.

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Table 2. approximated pH 's

$[H^+]$	100	80	64	50	40	32	25	20	16
approx	6.8	7.0	7.2	7.3	7.4	7.5	7.6	7.6	7.6

Let's find a better approximation. As $10^{1.6} \sim 40$,

$$\begin{aligned} pH &= 9 - \log_{10} [H^+] = 7.4 + 1.6 - \log_{10} [H^+] = 7.4 + \log_{10} 10^{1.6} - \log_{10} [H^+] \\ &\sim 7.4 + \log_{10} 40 - \log_{10} [H^+] = 7.4 + \log_{10} \frac{40}{[H^+]} \\ &= 7.4 + \log_{10} e \cdot \ln \frac{40}{[H^+]} = 7.4 + 0.4343 \cdot \ln \frac{40}{[H^+]} \end{aligned}$$

By the new approximation of the natural logarithm,

$$pH \sim 7.4 + 0.4343 \cdot \frac{40 - [H^+]}{M(40, [H^+])}$$

Let's substitute $M(40, [H^+])$ by $\sqrt{40[H^+]}$. Then we have

$$pH \sim 7.4 + 0.4343 \cdot \frac{40 - [H^+]}{\sqrt{40[H^+]}}$$

$$\therefore pH \sim 7.4 + 0.4343 \cdot \left(\sqrt{\frac{40}{[H^+]}} - \sqrt{\frac{[H^+]}{40}} \right)$$

Let $[H^+] = 40$ nanomol/l. Then we have $pH = 7.40$.

Let x be $[H^+] - 40$. Then we have

$$pH \sim 7.4 + 0.4343 \cdot \ln \frac{40}{40+x}$$

and

$$pH \sim 7.4 + 0.4343 \cdot \left(\sqrt{\frac{40}{40+x}} - \sqrt{\frac{40+x}{40}} \right)$$

Both graphs are shown in Figure 7 below. They are so close that it is very difficult to distinguish them. It tells us how accurate this approximation is.

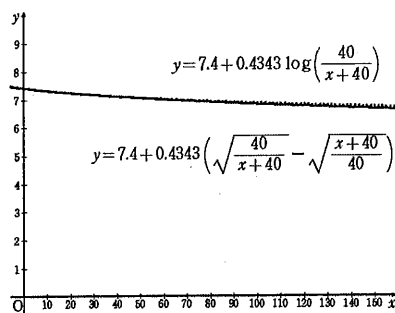


Figure 7. new approximation of pH

This approximation can show us why the hydrogen ion concentration and pH relate inversely. Suppose that $[H^+]$ increase. Then $\sqrt{\frac{40}{[H^+]}}$ decrease and $\sqrt{\frac{[H^+]}{40}}$ increase. So the pH decrease. On the other hand, suppose that $[H^+]$ decrease. Then $\sqrt{\frac{40}{[H^+]}}$ increase and $\sqrt{\frac{[H^+]}{40}}$ decrease. So the pH increase.

This relationship is called “ pH Scale Principle”.

In hospital, arterial blood gas test is commonly used to see patient’s condition. In human body, the acid-base balance is maintained by many systems. One of them is the carbonic acid-bicarbonate buffer system. The equilibrium



is helping to maintain the blood pH at a normal lever between 7.35 and 7.45.

Let PCO_2 and $[HCO_3^-]$ denote the carbon dioxide partial pressure and HCO_3^- ion concentration respectively. Then by the Henderson-Hasselbalch equation,

$$\begin{aligned} pH &= 6.10 + \log_{10} \frac{[HCO_3^-]}{0.03 PCO_2} = 6.10 + \log_{10} \left(\frac{100}{3} \cdot \frac{[HCO_3^-]}{PCO_2} \right) \\ &= 6.10 + \log_{10} 20 + \log_{10} \frac{5[HCO_3^-]}{3 PCO_2} = 7.40 + \log_{10} e \cdot \ln \frac{5[HCO_3^-]}{3 PCO_2} \\ \therefore pH &\sim 7.40 + 0.4343 \cdot \frac{5[HCO_3^-] - 3 PCO_2}{M(5[HCO_3^-], 3 PCO_2)} \end{aligned}$$

Let’s substitute $M(5[HCO_3^-], 3 PCO_2)$ by $\sqrt{5[HCO_3^-] \cdot 3 PCO_2}$. Then we have

$$\begin{aligned} pH &\sim 7.40 + 0.4343 \cdot \frac{5[HCO_3^-] - 3 PCO_2}{\sqrt{5[HCO_3^-] \cdot 3 PCO_2}} \\ \therefore pH &\sim 7.4 + 0.4343 \cdot \left(\sqrt{\frac{5[HCO_3^-]}{3 PCO_2}} - \sqrt{\frac{3 PCO_2}{5[HCO_3^-]}} \right) \end{aligned}$$

For example, suppose that a patient’s blood gas test indicates $PCO_2 = 48$ (Torr) and $[HCO_3^-] = 24$ (mEq/l). $[HCO_3^-]$ is in the normal range, but the value of PCO_2 is too high. Let’s calculate patient’s pH .

$$pH \sim 7.4 + 0.4343 \cdot \left(\sqrt{\frac{5 \cdot 24}{3 \cdot 48}} - \sqrt{\frac{3 \cdot 48}{5 \cdot 24}} \right) = 7.4 + 0.4343 \left(\sqrt{\frac{5}{6}} - \sqrt{\frac{6}{5}} \right) = 7.32$$

As the pH is smaller than 7.35, we can conclude that the patient is acidotic.

References

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2. Hirokazu Imai(2007). Acid Base Equilibrium, Water, and Electrolyte, Yodosya